Target Position Localization in a Passive Radar System Through Convex Optimization *

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ABSTRACT

This paper proposes efficient target localization methods for a passive radar system using bistatic time-of-arrival (TOA) information measured at multiple synthetic array locations, where the position of these synthetic array locations is subject to random errors. Since maximum likelihood (ML) formulation of this target localization problem is a non-convex optimization problem, semi-definite relaxation (SDR)-based optimization methods in general do not provide satisfactory performance. As a result, approximated ML optimization problems are proposed and solved with SDR plus bisection methods. For the case without position errors, it is shown that the relaxation guarantees a rank-one solution. The optimization problem for the case with position errors involves only a relaxation of a scalar quadratic term. Simulation results show that the proposed algorithms outperform existing methods and provide mean square position error performance very close to the Cramer-Rao lower bound even for larger values of noise and position estimation errors.

1. INTRODUCTION

In recent years, bistatic passive radar (BPR) systems, which utilize broadcast signals as sources of opportunity, have attracted significant interests due to their low cost, covertness, and availability of rich illuminator sources [1–4]. Compared to conventional active radar systems which typically operate in a monostatic mode and emit stronger signals with a wide signal bandwidth, BPR systems use broadcast signals which in general are very weak and have an extremely narrow bandwidth. From a target localization perspective, these features make it difficult to accurately estimate target positions exploiting a BPR system. In addition, BPR receivers may often be implemented on aerial or ground moving vehicles. In this case, the radar platform may only have inaccurate knowledge about its own instantaneous position. This uncertainty is caused by the accuracy limitation of the positioning system as well as multipath propagations.

Target localization is an important task that received extensive studies in various applications, such as wireless communications, sensor networks, urban canyon, and through-the-wall radar systems [5–8]. Specifically, multi-lateration techniques utilize the range information observed at multiple positions, which are distributed over a region, to uniquely localize a target. Depending on the applications, range information can be obtained using time-of-arrival (TOA), time-delay-of-arrival (TDOA), and received signal strength indicator (RSSI). On the other hand, the observation positions may be achieved using fixed receivers, or synthesized using a single moving platform. In the latter case, the receiver positions are subject to inaccuracy.

In all these applications, maximum likelihood (ML) estimation is considered as a powerful method of estimating the targets' location, which in general is a non-convex optimization problem. When the measurement noise is sufficiently small, the ML estimation problem may be solved using linearized least squares (LLS) estimation methods [5, 7]. The key steps of the LLS estimation methods are linearizing the objective function using Taylor's series expansion at some initial guess of target position and updating it with the least squares (LS) solution in an iterative approach. Like in many iterative optimization techniques for non-convex problems, however, the accuracy of the LLS estimator highly depends on the initial guess of the target's location. This has motivated researchers to consider more efficient designs. One such approach is the semi-definite relaxation

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(SDR) technique [6, 9–11], which converts a non-convex optimization problem into a convex one by relaxing certain rank constraints. It is worthwhile to mention that SDR-based approaches outperform computationally efficient two-step weighted least squares method proposed in [12], especially when the noise level is high and the sensor positions are not perfectly known.

The accuracy (or tightness) of SDR techniques, however, is problem specific, as shown in [13] for the TOA based optimization problems. For example, in optimization problems based on TDOA [9] and TOA [10], where an unknown time instant of the source's signal transmission is also optimized, SDR relaxations may not be tight and, thus, the penalty function approach is introduced. This is true also for robust designs where sensor positions are subject to certain random errors [9, 10]. In this context, the authors in [14] propose to use an approximate ML function in the SDR-based source localization problem, where the main idea is to neglect the second-order terms of noise signal [12].

In this paper, we propose an approximated ML estimation approach for target localization in a BPR system using bistatic TOA information measured at different locations of a moving receiver. As we discussed above, the range resolution is poor because of the narrow signal bandwidth and weak signal levels, and the receiver positions are subject to inaccurate knowledge of their own positions. Therefore, an optimization problem is also formulated for the case where the receiver positions are subject to estimation errors. The underlying optimization problems are still non-convex, but can be reformulated as convex problems using SDR and solved in conjunction with the bisection method. When there are no position errors, the SDR is confirmed to provide a rank-one solution. When there are estimation errors, the corresponding optimization problem involves only a relaxation of a quadratic scalar term.

The target localization problem and optimization technique described in this paper differ from existing literature in a number of ways. In contrast to the optimization problems in [6] and [13], where the objective function is solely a function of monostatic range, the objective function in our case involves bistatic range, which makes accurate target position estimation much more challenging. Further, in contrast to [9] and [10], where SDR of several variables and penalty function approach are employed, our approach involves SDR of only one variable, though in conjunction with the bisection approach. It is worthwhile to emphasize that the SDR for perfectly known positions guarantees rank-one solutions, whereas the optimization problem for imperfectly known positions involves relaxation of only a scalar quadratic term. For these reasons, the proposed method provides better performance than those of [9, 10] and does not require refinement through local optimization.

Notations: Upper (lower) bold face letters will be used for matrices (vectors); $(\cdot)^T$, $\mathbf{I}_n$, $||\cdot||$, $\mathrm{tr}(\cdot)$, and $\mathbf{A} \succeq 0$ denote transpose, $n \times n$ identity matrix, Euclidean norm, matrix trace operator, and positive semi-definiteness of $\mathbf{A}$, respectively.

2. SYSTEM MODEL

Consider a BPR system in which a moving receiver observes the direct signal from an illuminator and the reflected signal from a target at M different positions. The TOA of the direct signal from the illuminator at the ith receiver position, where $1 \leq i \leq M$, is given by

$$\tau_{\mathrm{d},i} = \frac{1}{c}||\mathbf{t} - \tilde{\mathbf{r}}_i||, \tag{1}$$

where c is the speed of light, $\mathbf{t}$ and $\tilde{\mathbf{r}}_i$ are column vectors of length n that represent, respectively, the coordinates of the illuminator and the receiver at the ith position. Depending on applications, n is 2 for a two-dimensional coordinate system and 3 for a three-dimension coordinate system. $\mathbf{t}$ is assumed to be stationary and precisely known. The TOA of the target reflected signal at the ith receiver position is given by

$$\tau_{\mathrm{b},i} = \frac{1}{c}\left\{||\mathbf{t} - \mathbf{p}|| + ||\mathbf{p} - \tilde{\mathbf{r}}_i||\right\}, \tag{2}$$

where $\mathbf{p}$ is the $n \times 1$ vector representing the location information of the target. By correlating the target reflected signal and the direct path, the effective TDOA between $\tau_{\mathrm{d},i}$ and $\tau_{\mathrm{b},i}$ can be estimated as

$$\bar{\tau}_i = \tau_{\mathrm{b},i} - \tau_{\mathrm{d},i} + \bar{n}_i = \frac{1}{c}\left[||\mathbf{t} - \mathbf{p}|| + ||\mathbf{p} - \tilde{\mathbf{r}}_i|| - ||\mathbf{t} - \tilde{\mathbf{r}}_i||\right] + \bar{n}_i, \tag{3}$$

where $\bar{n}_i$ is the measurement noise, which is assumed to be zero-mean Gaussian distributed. For notational simplicity, we denote $\tau_i = c\bar{\tau}_i$ and $n_i = c\bar{n}_i$. In this case, the ML estimate of the target location is expressed as

$$\hat{\mathbf{p}} = \arg\min_{\mathbf{p}} \sum_{i=1}^{M} \left[\tilde{\tau}_i - ||\mathbf{t} - \mathbf{p}|| - ||\mathbf{p} - \tilde{\mathbf{r}}_i||\right]^2, \tag{4}$$

where

$$\tilde{\tau}_i = \tau_i + ||\mathbf{t} - \tilde{\mathbf{r}}_i||. \tag{5}$$

Note that the knowledge of the true position of the ith receiver, $\tilde{\mathbf{r}}_i$, may be inaccurate. Denote the estimated receiver position by $\mathbf{r}_i$. The relationship between $\tilde{\mathbf{r}}_i$ and $\mathbf{r}_i$ is expressed as

$$\tilde{\mathbf{r}}_i = \mathbf{r}_i + \mathbf{e}_i \tag{6}$$

where $\mathbf{e}_i$ is the random estimation error for the ith receiver position.

3. PROPOSED OPTIMIZATION APPROACHES

The unconstrained minimization problem (4) is non-convex. Thus, it is difficult to obtain the global optimum solution with a reasonable complexity. The SDR-based optimization methods applied in [9,10] may not be tight enough in general and, thus, may often fail to provide performance sufficiently close to CRLB. In this section and inspired from [12], we propose an alternative approach that approximates the ML target localization problem. However, this approximate ML problem is shown to be less sensitive to the SDR and perform better than the methods that solve the exact ML problem.

3.1 Optimization without Receiver Position Error

We first consider the case where the receiver positions are exactly known, i.e., $\tilde{\mathbf{r}}_i = \mathbf{r}_i$. Denote $\tau_i = \tau_i^{\text{o}} + n_i$, where τ_i^{o} is the noise-free observation. Then, from (7), we have

$$\tau_i^{\text{o}} = ||\mathbf{t} - \mathbf{p}|| + ||\mathbf{r}_i - \mathbf{p}|| - ||\mathbf{t} - \mathbf{r}_i|| \tag{7}$$

or equivalently

$$\tilde{\tau}_i^{\text{o}} - ||\mathbf{t} - \mathbf{p}|| = ||\mathbf{r}_i - \mathbf{p}||, \tag{8}$$

where $\tilde{\tau}_i^{\text{o}} = \tau_i^{\text{o}} + ||\mathbf{t} - \mathbf{r}_i||$. Squaring both sides of (8) and substituting $\tilde{\tau}_i^{\text{o}} = \tilde{\tau}_i - n_i$ into (8), we obtain

$$\tilde{\tau}_i^2 - 2(\tilde{\tau}_i - q)n_i + n_i^2 - 2\tilde{\tau}_i q + ||\mathbf{t}||^2 - ||\mathbf{r}_i||^2 - 2(\mathbf{t} - \mathbf{r}_i)^T\mathbf{p} = 0, \tag{9}$$

where $q = ||\mathbf{t} - \mathbf{p}||$. Neglecting the second-order terms of the noise, n_i^2, and stacking (9) for all i, we obtain the following expression

$$\mathbf{u} - \mathbf{B}\mathbf{z} \approx \mathbf{D}\mathbf{n}, \tag{10}$$

where

$$\begin{aligned}
\mathbf{u} &= \left[||\mathbf{t}||^2 - ||\mathbf{r}_1||^2 + \tilde{\tau}_1^2, \cdots, ||\mathbf{t}||^2 - ||\mathbf{r}_M||^2 + \tilde{\tau}_M^2\right]^T, \\
\mathbf{B} &= 2\begin{bmatrix} (\mathbf{t} - \mathbf{r}_1)^T & \tilde{\tau}_1 \\ (\mathbf{t} - \mathbf{r}_2)^T & \tilde{\tau}_2 \\ \vdots & \vdots \\ (\mathbf{t} - \mathbf{r}_M)^T & \tilde{\tau}_M \end{bmatrix}, \\
\mathbf{D} &= 2\begin{bmatrix} \tilde{\tau}_1 - q & 0 & \cdots & 0 \\ 0 & \tilde{\tau}_2 - q & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \tilde{\tau}_M - q \end{bmatrix}, \\
\mathbf{z} &= [\mathbf{p}^T, q]^T, \\
\mathbf{n} &= [n_1, n_2, \cdots, n_M]^T.
\end{aligned} \tag{11}$$

Note that $\mathbf{n}$ is a vector of zero-mean i.i.d. Gaussian random variables of variance σ_{n}^2. From (10), the noise vector can be approximated as

$$\mathbf{n} \approx \mathbf{D}^{-1}(\mathbf{u} - \mathbf{B}\mathbf{z}). \tag{12}$$

As a result, the ML target localization problem can be approximated as the following minimization problem

$$\begin{aligned} &\min_{\mathbf{z}} \frac{1}{\sigma_{\mathrm{n}}^2}(\mathbf{u} - \mathbf{B}\mathbf{z})^T \mathbf{D}^{-1}\mathbf{D}^{-1}(\mathbf{u} - \mathbf{B}\mathbf{z}) \\ &\text{s.t. } q \triangleq z(n+1) = ||\mathbf{t} - \mathbf{p}||, \end{aligned} \tag{13}$$

which, after omitting the constant scaling factor $\frac{1}{\sigma_{\mathrm{n}}^2}$, can be expressed as

$$\begin{aligned} &\min_{\mathbf{z}} \sum_{i=1}^{M} \frac{(u_i - \mathbf{b}_i^T \mathbf{z})^2}{(\tilde{\tau}_i - q)^2} \\ &\text{s.t. } q = ||\mathbf{t} - \mathbf{p}||, \end{aligned} \tag{14}$$

where u_i is the ith element of $\mathbf{u}$ and $\mathbf{b}_i^T$ is the ith row of $\mathbf{B}$. We define $\mathbf{p} \triangleq \tilde{\mathbf{z}} = [z(1), \cdots, z(n)]^T$, $\tilde{\mathbf{b}}_i^T = [b_i(1), \cdots, b_i(n)]$ and $v_i = b_i(n+1)$, where $b_i(k)$ is the kth element of $\mathbf{b}_i^T$. The objective function (13) can be expressed as

$$f_{\mathrm{ob}} = \sum_{i=1}^{M} \left[\frac{u_i}{\tilde{\tau}_i - q} - \frac{\tilde{\mathbf{b}}_i^T \tilde{\mathbf{z}}}{\tilde{\tau}_i - q} - \frac{v_i q}{\tilde{\tau}_i - q} \right]^2, \tag{15}$$

which, after some manipulations, can be expressed as

$$f_{\mathrm{ob}} = [\tilde{\mathbf{z}}^T, 1]\mathbf{G}(q)[\tilde{\mathbf{z}}^T, 1]^T \tag{16}$$

where the $(n+1) \times (n+1)$ matrix $\mathbf{G}(q)$ is given by

$$\mathbf{G}(q) = \begin{bmatrix} \sum_{i=1}^{M} \frac{\tilde{\mathbf{b}}_i \tilde{\mathbf{b}}_i^T}{(\tilde{\tau}_i - q)^2} & \sum_{i=1}^{M} \frac{\tilde{\mathbf{b}}_i (v_i q - u_i)}{(\tilde{\tau}_i - q)^2} \\ \sum_{i=1}^{M} \frac{\tilde{\mathbf{b}}_i^T (v_i q - u_i)}{(\tilde{\tau}_i - q)^2} & \sum_{i=1}^{M} \frac{(v_i q - u_i)^2}{(\tilde{\tau}_i - q)^2} \end{bmatrix}. \tag{17}$$

Thus, the minimization problem (14) is given by

$$\begin{aligned} &\min_{\tilde{\mathbf{z}}, q} \mathrm{tr}\left\{ \begin{bmatrix} \tilde{\mathbf{z}} \\ 1 \end{bmatrix} [\tilde{\mathbf{z}}^T, 1]\mathbf{G}(q) \right\} \\ &\text{s.t. } q^2 = ||\mathbf{t} - \mathbf{p}||^2 \longleftrightarrow q^2 = \mathrm{tr}\left\{ \begin{bmatrix} \tilde{\mathbf{z}} \\ 1 \end{bmatrix} [\tilde{\mathbf{z}}^T, 1] \begin{bmatrix} \mathbf{I}_n & \mathbf{t} \\ -\mathbf{t}^T & \mathbf{t}^T\mathbf{t} \end{bmatrix} \right\}, \end{aligned} \tag{18}$$

which is clearly a non-convex optimization problem. We define $\tilde{\mathbf{Z}} = \tilde{\mathbf{z}}\tilde{\mathbf{z}}^T$, which is relaxed as $\tilde{\mathbf{Z}} \succeq \tilde{\mathbf{z}}\tilde{\mathbf{z}}^T$, i.e., $\tilde{\mathbf{Z}} - \tilde{\mathbf{z}}\tilde{\mathbf{z}}^T$ is positive semi-definite. This yields the following optimization problem

$$\begin{aligned} &\min_{\tilde{\mathbf{z}}, \tilde{\mathbf{Z}}, q} \mathrm{tr}\left\{ \begin{bmatrix} \tilde{\mathbf{Z}} & \tilde{\mathbf{z}} \\ \tilde{\mathbf{z}}^T & 1 \end{bmatrix} \mathbf{G}(q) \right\} \\ &\text{s.t. } q^2 = \mathrm{tr}\left\{ \begin{bmatrix} \tilde{\mathbf{Z}} & \tilde{\mathbf{z}} \\ \tilde{\mathbf{z}}^T & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I}_n & \mathbf{t} \\ -\mathbf{t}^T & \mathbf{t}^T\mathbf{t} \end{bmatrix} \right\}, \\ &\begin{bmatrix} 1 & \tilde{\mathbf{z}}^T \\ \tilde{\mathbf{z}} & \tilde{\mathbf{Z}} \end{bmatrix} \succeq 0. \end{aligned} \tag{19}$$

For a given q, the above optimization problem is a convex function of $\tilde{\mathbf{z}}$ and $\tilde{\mathbf{Z}}$. However, the joint optimization over q, $\tilde{\mathbf{z}}$ and $\tilde{\mathbf{Z}}$ is not convex. Nevertheless, since $q \geq 0$ is a scalar variable, the joint optimization problem can be solved by using one dimensional (e.g., bisection) search with respect to q. Assume that the optimum q lies in the interval $[q_{\rm l}, q_{\rm u}]$. It is obvious in the underlying problem that $q_{\rm l} = 0$. The algorithm (Algorithm 1) for solving (19) is then provided below.

- 1) Initialize $q_{\rm l}$, $q_{\rm u}$ and set $\epsilon > 0$.
- 2) Solve (19) with $q = q_{\rm l}$ and $q = q_{\rm u}$.
- 3) If $f_{\rm ob}(q_{\rm l}) < f_{\rm ob}(q_{\rm u})$, set $q_{\rm u} = \frac{q_{\rm l}+q_{\rm u}}{2}$; otherwise set $q_{\rm l} = \frac{q_{\rm l}+q_{\rm u}}{2}$.
- 4) Go to step 2 until $|q_{\rm u} - q_{\rm l}| \leq \epsilon$.

Remark 1: Let $[\tilde{\mathbf{z}}^*, \tilde{\mathbf{Z}}^*, q^*]$ be an optimal solution of the problem (19). Notice that the number of equality constraints is $L = 1$. According to Shapiro-Barvinok-Pataki (SBP) result [15], there exists an optimal solution $\tilde{\mathbf{Z}}^*$ such that $\mathrm{rank}(\tilde{\mathbf{Z}}^*)(\mathrm{rank}(\tilde{\mathbf{Z}}^*) + 1) \leq 2L$. Since $L = 1$ in (19) and $\mathrm{rank}(\tilde{\mathbf{Z}}^*) \neq 0$, we find that $\tilde{\mathbf{Z}}^*$ is rank-one.

3.2 Optimization with Receiver Position Error

In the presence of random position errors (i.e., $\tilde{\mathbf{r}}_i = \mathbf{r}_i + \mathbf{e}_i$), we can express the measured range difference for the ith receiver position as

$$\tau_i^{\rm r} = ||\mathbf{t} - \mathbf{p}|| + ||(\mathbf{r}_i + \mathbf{e}_i) - \mathbf{p}|| - ||(\mathbf{r}_i + \mathbf{e}_i) - \mathbf{t}|| + n_i. \tag{20}$$

We consider that $\mathbf{e}_i$ are small when compared to $\mathbf{r}_i - \mathbf{p}$ and $\mathbf{r}_i - \mathbf{t}$, i.e., $||\mathbf{e}_i|| << \{||\mathbf{r}_i - \mathbf{p}||, ||\mathbf{r}_i - \mathbf{t}||\}$. Using Taylor's series expansion, we get

$$\begin{aligned} ||(\mathbf{r}_i + \mathbf{e}_i) - \mathbf{p}|| &= ||\mathbf{r}_i - \mathbf{p}|| + \mathbf{e}_i^T \frac{\mathbf{r}_i - \mathbf{p}}{||\mathbf{r}_i - \mathbf{p}||} + \mathcal{O}(||\mathbf{e}_i||), \\ ||(\mathbf{r}_i + \mathbf{e}_i) - \mathbf{t}|| &= ||\mathbf{r}_i - \mathbf{t}|| + \mathbf{e}_i^T \frac{\mathbf{r}_i - \mathbf{t}}{||\mathbf{r}_i - \mathbf{t}||} + \mathcal{O}(||\mathbf{e}_i||), \end{aligned} \tag{21}$$

where $\mathcal{O}(||\mathbf{e}_i||)$ stands for higher order terms of $||\mathbf{e}_i||$. Substituting (21) into (20), we obtain

$$\tau_i^{\rm r} \approx ||\mathbf{t} - \mathbf{p}|| + ||\mathbf{r}_i - \mathbf{p}|| - ||\mathbf{r}_i - \mathbf{t}|| + \mathbf{e}_i^T \left[\frac{\mathbf{r}_i - \mathbf{p}}{||\mathbf{r}_i - \mathbf{p}||} - \frac{\mathbf{r}_i - \mathbf{t}}{||\mathbf{r}_i - \mathbf{t}||} \right] + n_i. \tag{22}$$

Define $\tilde{\tau}_i^{\rm r} = \tau_i^{\rm r} + ||\mathbf{r}_i - \mathbf{t}||$, $\tilde{\mathbf{w}}_i = \frac{\mathbf{r}_i - \mathbf{p}}{||\mathbf{r}_i - \mathbf{p}||}$ and $\mathbf{w}_i = \frac{\mathbf{r}_i - \mathbf{t}}{||\mathbf{r}_i - \mathbf{t}||}$. Then, (22) can be expressed as

$$(\tilde{\tau}_i^{\rm r} - n_i) + \mathbf{e}_i^T \mathbf{w}_i - ||\mathbf{t} - \mathbf{p}|| \approx ||\mathbf{r}_i - \mathbf{p}|| + \mathbf{e}_i^T \tilde{\mathbf{w}}_i. \tag{23}$$

Squaring both sides of (23) and after some derivations, we get

$$\begin{aligned} &(\tilde{\tau}_i^{\rm r})^2 + ||\mathbf{t}||^2 - ||\mathbf{r}_i||^2 - 2(\mathbf{t} - \mathbf{r}_i)^T \mathbf{p} - 2\tilde{\tau}_i^{\rm r} q + n_i^2 \approx \\ &2n_i(\tilde{\tau}_i^{\rm r} - q) + 2\mathbf{e}_i^T \left(\mathbf{r}_i - \mathbf{p} - \tilde{\tau}_i^{\rm r} \mathbf{w}_i + q\mathbf{w}_i \right) + 2n_i \mathbf{e}_i^T \mathbf{w}_i. \end{aligned} \tag{24}$$

Neglecting the second order terms of noise and the cross-order term between noise and position error, for $i = 1, \cdots, M$, (24) can be expressed in vector form as

$$\mathbf{u} - \mathbf{B}\mathbf{z} \approx \mathbf{D}\mathbf{n} + \mathbf{C}\mathbf{e}_{\rm L} \tag{25}$$

where

$$\begin{aligned} \mathbf{C} &= 2 \begin{bmatrix} \mathbf{c}_1^T & \mathbf{0}_{1\times n} & \cdots & \mathbf{0}_{1\times n} \\ \mathbf{0}_{1\times n} & \mathbf{c}_2^T & \cdots & \mathbf{0}_{1\times n} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{0}_{1\times n} & \mathbf{0}_{1\times n} & \cdots & \mathbf{c}_M^T \end{bmatrix}, \\ \mathbf{c}_i^T &= (\mathbf{r}_i - \mathbf{p})^T - \mathbf{w}_i^T (\tilde{\tau}_i^{\rm r} - q), \forall i, \\ \mathbf{e}_{\rm L} &= [\mathbf{e}_1^T, \mathbf{e}_2^T, \cdots, \mathbf{e}_M^T]^T. \end{aligned} \tag{26}$$

Notice that (25) can be expressed as

$$\mathbf{D}^{-1}(\mathbf{u}-\mathbf{B}\mathbf{z}) \approx \mathbf{n}+\mathbf{D}^{-1}\mathbf{C}\mathbf{e}_{\mathrm{L}}. \tag{27}$$

Assume that the position error corresponding to the ith receiver is Gaussian distributed with zero-mean and covariance matrix $\mathbf{Q}_i$. Assuming that $\{\mathbf{e}_i\}_{i=1}^M$ are statistically independent, we have $\mathbf{Q}_{\mathrm{L}} = \mathrm{E}\left\{\mathbf{e}_{\mathrm{L}}\mathbf{e}_{\mathrm{L}}^T\right\} = \mathrm{blkdiag}(\mathbf{Q}_1,\cdots,\mathbf{Q}_M)$, where blkdiag stands for a block-diagonal matrix. Moreover, since $\{\mathbf{e}_i\}_{i=1}^M$ and $\mathbf{n}_i$ are independent, we obtain

$$\mathrm{E}\left\{(\mathbf{n}+\mathbf{D}^{-1}\mathbf{C}\mathbf{e}_{\mathrm{L}})(\mathbf{n}+\mathbf{D}^{-1}\mathbf{C}\mathbf{e}_{\mathrm{L}})^T\right\} = \sigma_{\mathrm{n}}^2\mathbf{I}_M + \mathbf{D}^{-1}\mathbf{C}\mathbf{Q}_{\mathrm{L}}\mathbf{C}^T(\mathbf{D}^{-1})^T. \tag{28}$$

Particularly, under the assumption $\mathbf{Q}_i = \sigma_{\mathrm{e}}^2\mathbf{I}_n$, where σ_{e}^2 is the variance of the elements of $\{\mathbf{e}_i\}_{i=1}^M$, the ML localization problem can be approximately expressed as

$$\begin{aligned}&\min_{\mathbf{z}}\left\{(\mathbf{u}-\mathbf{B}\mathbf{z})^T\mathbf{D}^{-1}\left(\sigma_{\mathrm{n}}^2\mathbf{I}_M+\sigma_{\mathrm{e}}^2\mathbf{D}^{-1}\mathbf{C}\mathbf{C}^T\mathbf{D}^{-1}\right)^{-1}\mathbf{D}^{-1}(\mathbf{u}-\mathbf{B}\mathbf{z})\right\}\\ &\text{s.t. } q \triangleq z(n+1) = ||\mathbf{t}-\mathbf{p}||.\end{aligned} \tag{29}$$

The objective function of the optimization problem (29) can be simplified to

$$f_{\mathrm{rob}} = (\mathbf{u}-\mathbf{B}\mathbf{z})^T\left(\sigma_{\mathrm{n}}^2\mathbf{D}\mathbf{D}^T+\sigma_{\mathrm{e}}^2\mathbf{C}\mathbf{C}^T\right)^{-1}(\mathbf{u}-\mathbf{B}\mathbf{z}), \tag{30}$$

where

$$\begin{aligned}\mathbf{D}\mathbf{D}^T &= 4\begin{bmatrix}(\tilde{\tau}_1^{\mathrm{r}}-q)^2 & 0 & \cdots & 0\\ 0 & (\tilde{\tau}_2^{\mathrm{r}}-q)^2 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \cdots & (\tilde{\tau}_M^{\mathrm{r}}-q)^2\end{bmatrix},\\ \mathbf{C}\mathbf{C}^T &= \begin{bmatrix}\mathbf{c}_1^T\mathbf{c}_1 & 0 & \cdots & 0\\ 0 & \mathbf{c}_2^T\mathbf{c}_2 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \cdots & \mathbf{c}_M^T\mathbf{c}_M\end{bmatrix}.\end{aligned} \tag{31}$$

Substituting (31) into (30), f_{rob} can be expressed as

$$f_{\mathrm{rob}} = \frac{1}{4}\sum_{i=1}^M \frac{(u_i-\mathbf{b}_i^T\mathbf{z})^2}{\sigma^2(\tilde{\tau}_i^{\mathrm{r}}-q)^2+\sigma_{\mathrm{e}}^2\left(||\mathbf{r}_i-\mathbf{p}||^2+2(\tilde{\tau}_i-q)(\mathbf{r}_i-\mathbf{p})^T\mathbf{w}_i\right)}, \tag{32}$$

where $\sigma^2 = \sigma_{\mathrm{e}}^2+\sigma_{\mathrm{n}}^2$. With the following definitions

$$\begin{aligned}\bar{d}_i &= \sigma^2(\tilde{\tau}_i^{\mathrm{r}})^2+\sigma_{\mathrm{e}}^2\left(||\mathbf{r}_i||^2-2\tilde{\tau}_i^{\mathrm{r}}\mathbf{r}_i^T\mathbf{w}_i\right),\\ \bar{f}_i &= 2\left(\sigma_{\mathrm{e}}^2\mathbf{r}_i^T\mathbf{w}_i-\sigma^2\tilde{\tau}_i^{\mathrm{r}}\right),\\ \mathbf{g}_i^T &= 2\sigma_{\mathrm{e}}^2\left(\tilde{\tau}_i^{\mathrm{r}}\mathbf{w}_i^T-\mathbf{r}_i^T\right),\end{aligned} \tag{33}$$

the denominator and numerator of the ith term of f_{rob}, respectively, can be given by

$$\begin{aligned}f_{\mathrm{den},i} &= \bar{d}_i+\bar{f}_iq+\mathbf{g}_i^T\tilde{\mathbf{z}}+\sigma^2q^2-2\sigma_{\mathrm{e}}^2q\tilde{\mathbf{z}}^T\mathbf{w}_i+\sigma_{\mathrm{e}}^2\tilde{\mathbf{z}}^T\tilde{\mathbf{z}},\\ f_{\mathrm{num},i} &= (u_i-\tilde{\mathbf{b}}_i^T\tilde{\mathbf{z}}-v_iq)^2.\end{aligned} \tag{34}$$

Introducing the auxiliary variables $t_i \geq 0, \forall i$, the optimization problem (29) is then expressed as

$$\begin{aligned}&\min_{t_i,q,\tilde{\mathbf{z}}}\sum_{i=1}^M t_i\\ &\text{s.t. } f_{\mathrm{den},i} \geq \frac{(u_i-\tilde{\mathbf{b}}_i^T\tilde{\mathbf{z}}-v_iq)^2}{t_i}, \forall i\\ &\quad q^2 = ||\mathbf{t}-\tilde{\mathbf{z}}||^2.\end{aligned} \tag{35}$$

Using the Schur-complement theorem [18], the ith inequality constraint of (35) can be expressed as

$$\begin{bmatrix} t_i & u_i - \tilde{\mathbf{b}}_i^T\tilde{\mathbf{z}} - v_i q \\ u_i - \tilde{\mathbf{b}}_i^T\tilde{\mathbf{z}} - v_i q & f_{\text{den},i} \end{bmatrix} \succeq 0. \tag{36}$$

Defining $\bar{z}_\text{s} = \tilde{\mathbf{z}}^T\tilde{\mathbf{z}}$ and using the relaxation $\bar{z}_\text{s} \geq \tilde{\mathbf{z}}^T\tilde{\mathbf{z}}$, the optimization problem (35) is given by

$$\begin{aligned} &\min_{t_i,q,\tilde{\mathbf{z}}} \sum_{i=1}^{M} t_i \\ &\text{s.t.} \begin{bmatrix} t_i & u_i - \tilde{\mathbf{b}}_i^T\tilde{\mathbf{z}} - v_i q \\ u_i - \tilde{\mathbf{b}}_i^T\tilde{\mathbf{z}} - v_i q & \tilde{f}_{\text{den},i} \end{bmatrix} \succeq 0, \forall i, \\ &\quad q^2 = ||\mathbf{t}||^2 - 2\mathbf{t}^T\tilde{\mathbf{z}} + \bar{z}_\text{s}, \\ &\quad \begin{bmatrix} 1 & \tilde{\mathbf{z}} \\ \tilde{\mathbf{z}}^T & \bar{z}_\text{s} \end{bmatrix} \succeq 0, \end{aligned} \tag{37}$$

where

$$\tilde{f}_{\text{den},i} = \bar{d}_i + \bar{f}_i q + \mathbf{g}_i^T\tilde{\mathbf{z}} + \sigma^2 q^2 - 2\sigma_\text{e}^2 q\tilde{\mathbf{z}}^T\mathbf{w}_i + \sigma_\text{e}^2\bar{z}_\text{s}. \tag{38}$$

For a given q, the optimization problem (37) is convex. The joint optimization is then solved in conjunction with the bisection search over q. The algorithm (Algorithm 1) as shown for the case without position errors can then be applied to solve (37).

4. COMPLEXITY ANALYSIS

We present the computational complexity of the proposed optimization algorithms using the approach [16]. For a given q, the number of iterations required for solving (19) is upper bounded by $\tilde{\mathcal{O}}\left((n+1)^{\frac{1}{2}}\right)$, whereas the work load per iteration is upper bounded by $\tilde{\mathcal{O}}\left((n^2+n)^2((n+1)^2+1)\right)$. The bisection search w.r.t. q requires $\bar{L} = \log_2\left(\frac{q_\text{u}-q_\text{l}}{\epsilon}\right)$ iterations. This means the overall complexity for the case withour position errors is approximately $\tilde{\mathcal{O}}\left(n^{6.5}\bar{L}\right)$. It is interesting to note that the complexity of Algorithm 1 does not depend on M. In a similar manner, for a given q, we can show that the complexity of (37) in terms of number of iterations is $\tilde{\mathcal{O}}\left((n+1+2M)^{\frac{1}{2}}\right)$, whereas the complexity per iteration is $\tilde{\mathcal{O}}\left((n+M)^2((n+1)^2+4M+1)\right)$. This means that the total complexity of (37) is approximately $\tilde{\mathcal{O}}\left((n+2M)^{0.5}(n+M)^2(n^2+4M)\right)$. For the case without position errors, the SDP approach (before local optimization) of [9] requires $\tilde{\mathcal{O}}\left((n+2+M)^{\frac{1}{2}}\right)$ iterations, where the complexity per-iteration is given by $\tilde{\mathcal{O}}\left((M^2+M+n+1)^2((M+1)^2+(n+1)^2+M^2)\right)$. For a gradient-based local optimization, such as steepest-descent method, $\tilde{\mathcal{O}}(\epsilon^{-2})$ iterations are required to keep the norm of the gradient below ϵ [17]. Thus, the total complexity is approximately given by $\tilde{\mathcal{O}}\left((n+M)^{0.5}(M^2+M+n)^2(2M^2+n^2)\right) + \tilde{\mathcal{O}}(\epsilon^{-2})$. On the other hand, for the case with position errors, the SDP approach (before local optimization) in [9] requires $\tilde{\mathcal{O}}\left((n+1+2(M+1))^{\frac{1}{2}}\right)$ iterations. The computational complexity of each iteration is given by $\tilde{\mathcal{O}}\left((M^2+M+n+3)^2\left(2(M+1)^2+(n+1)^2+M^2\right)\right)$, which results in a total approximate complexity of $\tilde{\mathcal{O}}\left((n+2M)^{0.5}(M^2+M+n)^2(3M^2+n^2)\right) + \tilde{\mathcal{O}}(\epsilon^{-2})$. Note that in practice $n << M$. For this case, the total complexity between the proposed and SDP methods is compared in Table 1.

5. NUMERICAL RESULTS AND DISCUSSIONS

Computer simulations are conducted to demonstrate the effectiveness of the proposed method. We consider that the receiver of the BPR system is mounted on a ground vehicle. As such, we use a two-dimensional coordinate system. As shown in Fig. 1, the stationary target and illuminators are located at positions $\mathbf{p} = [600, 550]^T$ meters

Method	Type	Complexity ($n << M$)
Proposed	Without errors, problem (19)	$\tilde{\mathcal{O}}\left(n^{6.5}\bar{L}\right)$
SDP	Without errors [9]	$\tilde{\mathcal{O}}\left(M^{6.5}\right)+\tilde{\mathcal{O}}\left(\epsilon^{-2}\right)$
Proposed	With errors, problem (37)	$\tilde{\mathcal{O}}\left(M^{3.5}\bar{L}\right)$
SDP	With errors [9]	$\tilde{\mathcal{O}}\left(M^{6.5}\right)+\tilde{\mathcal{O}}\left(\epsilon^{-2}\right)$

Table 1. Comparison of complexity between different methods.

and $\mathbf{t} = [-200, 0]^T$ meters, respectively. The ground vehicle moves around the target, first in the x-axis direction and then in the y-axis. The TOA information is measured at the following seven positions of the receiver,

$$\begin{bmatrix} 800 & 900 & 1000 & 1100 & 1100 & 1100 & 1100 \\ 200 & 200 & 200 & 200 & 300 & 400 & 500 \end{bmatrix} \text{meters.}$$

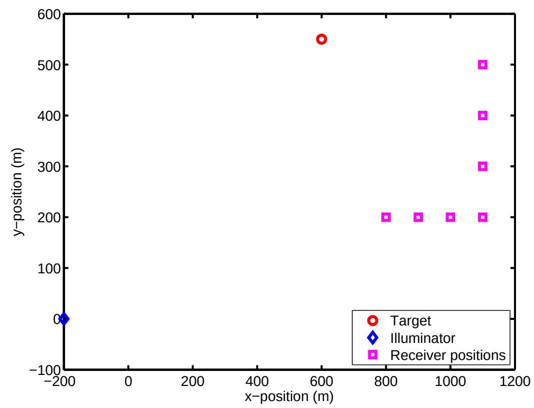

Figure 1. Scene of the simulations

The CVX toolbox [19] is used to solve the convex optimization problems (19) and (37). These optimization problems are solved within a framework of bisection algorithm outlined in Algorithm 1. We run Algorithm 1 and corresponding algorithm for (37) by taking $q_{\mathrm{l}} = 0$m, $q_{\mathrm{u}} = 1000$m and $\epsilon = 1$m. Note that smaller values of ϵ can be taken for improving convergence accuracy, whereas larger values of q_{u} can be taken if we do not have even a coarse knowledge of the illuminator-target range. Both settings in general result in higher computational complexity, since the bisection search requires more iterations. We compare the proposed method with the SDP method that employs local optimization [9] and the LLS method. Notice that the SDP method [9] without local optimization gives very poor results in our simulation scenarios, and thus, only the results after local optimization are shown. As suggested in [9], the penalty parameter for this method is varied between 10^{-3} and 10^{-7}. The solution of the LLS method at a given estimate $\mathbf{p}_0$ is given by

$$\mathbf{p} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T(\boldsymbol{\tau}^{\mathrm{mo}} - \mathbf{h}), \tag{39}$$

where

$$\begin{aligned}
\mathbf{A} &= \begin{bmatrix} \mathbf{a}_1^T \triangleq \dfrac{(\mathbf{p}_0-\mathbf{t})^T}{||\mathbf{p}_0-\mathbf{t}||} + \dfrac{(\mathbf{p}_0-\mathbf{r}_1)^T}{||\mathbf{p}_0-\mathbf{r}_1||} \\ \vdots \\ \mathbf{a}_M^T \triangleq \dfrac{(\mathbf{p}_0-\mathbf{t})^T}{||\mathbf{p}_0-\mathbf{t}||} + \dfrac{(\mathbf{p}_0-\mathbf{r}_M)^T}{||\mathbf{p}_0-\mathbf{r}_M||} \end{bmatrix}, \\
\mathbf{h} &= \begin{bmatrix} ||\mathbf{p}_0-\mathbf{t}|| + ||\mathbf{p}_0-\mathbf{r}_1|| - ||\mathbf{t}-\mathbf{r}_1|| + \mathbf{p}_0^T\mathbf{a}_1 \\ \vdots \\ ||\mathbf{p}_0-\mathbf{t}|| + ||\mathbf{p}_0-\mathbf{r}_M|| - ||\mathbf{t}-\mathbf{r}_M|| + \mathbf{p}_0^T\mathbf{a}_M \end{bmatrix}, \\
\boldsymbol{\tau}^{\mathrm{mo}} &= [\tau_1, \cdots, \tau_M]^T \text{ or } [\tau_1^{\mathrm{r}}, \cdots, \tau_M^{\mathrm{r}}]^T.
\end{aligned} \tag{40}$$

For the LLS method, we take an initial estimate as $\mathbf{p}_0 = [10, 10]^T$ and update $\mathbf{p}$ using (39) until $||\mathbf{p} - \mathbf{p}_0|| \le 1$. We first consider the case in which the observed data set is contaminated by measurement noise, whereas the receiver positions are perfectly known. The measurement noise follows a zero-mean Gaussian distribution with a variance of σ_{n}^2.

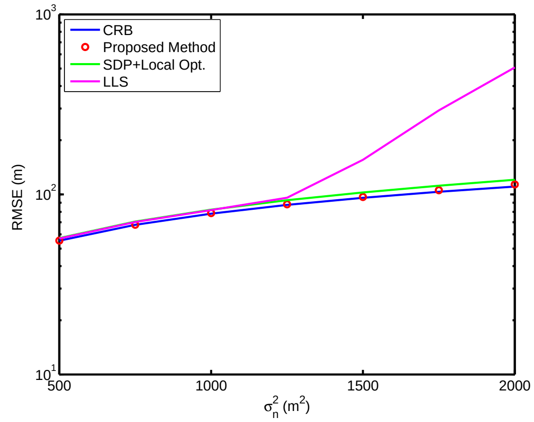

Figure 2. RMSE performance versus noise variance (no position error)

Fig. 2 shows the root-mean-square error (RMSE) of the estimated target position as a function of σ_{n}^2. The performance of the proposed method is compared with the LLS and SDP methods, and the Cramer-Rao lower bound (CRLB). It is observed from this figure that the proposed method significantly outperforms the LLS method and slightly outperforms the SDP method, especially for larger values of σ_{n}^2. The performance of the proposed method is very close to the CRLB for all σ_{n}^2. When $\sigma_{\mathrm{n}}^2 > 1250\mathrm{m}^2$, the RMSE performance of the LLS method deviates significantly from the CRLB, whereas that of the other two methods remains stable.

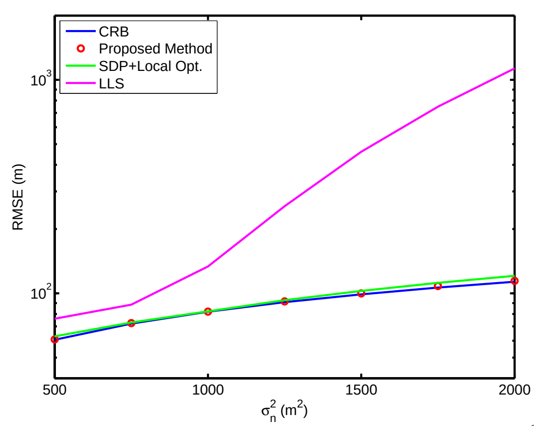

Figure 3. RMSE performance versus noise variance with $\sigma_{\mathrm{e}}^2 = 100\mathrm{m}^2$.

In Fig. 3, the performance of the proposed robust method is displayed for the case where the receiver positions are subject to i.i.d distributed Gaussian random errors with a variance of $\sigma_{\mathrm{e}}^2 = 100\mathrm{m}^2$. It can be observed from this figure that the performance of the LLS method degrades drastically when σ_{n}^2 increases. Both the proposed and the SDP methods provide performances close to the CRLB. However, the proposed method slightly outperforms the SDP method which employs local optimization.

6. CONCLUSIONS

We investigated the problem of localizing a target using time-of-arrival information measured at different locations of a receiver in a bistatic passive radar system. The localization problems are formulated using an approximate maximum likelihood (ML) estimate of the target location. The resulting non-convex problems are reformulated as convex problems using the semi-definite relaxation approach and solved in a framework of bisection algorithm. The optimization problems are examined for the cases when only measurement errors are present and when both the measurement and receiver position errors are present. Simulation results verify that the performance of the proposed methods is much closer to the CRLB and better than the SDP method.

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