Single-Snapshot Adaptive Beamforming

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***Abstract*—Adaptive beamformers are sensitive to model mismatch, especially when the number of training samples is small or the training samples are contaminated by the signal component. In this paper, we consider an extreme scenario where only a single signal-contaminated snapshot is available for adaptive beamformer design. In such a case, we cannot perform direct inversion or eigen-decomposition of the rank-one sample covariance matrix required in conventional adaptive beamformer design. To address this issue, we formulate a sparsity-constrained covariance matrix fitting problem to estimate the spatial spectrum distribution over the observed spatial domain, which is then used for adaptive beamformer design via the sparse reconstruction of the interference-plus-noise covariance matrix. Simulation results demonstrate the performance advantage of the proposed adaptive beamforming algorithm over other beamforming algorithms suitable for the single-snapshot scenario.**

***Index Terms*—Adaptive beamforming, covariance matrix fitting, covariance matrix reconstruction, single snapshot, sparsity.**

I. Introduction

Adaptive beamforming, as an effective array processing technique for spatial filtering, has been widely applied in various areas, such as radar, sonar, wireless communications, radio astronomy, speech processing, and medical imaging [1–6]. Compared with data-independent beamformers, adaptive beamformers provide better interference suppression capability by calculating the beamforming weight vector from the array received data. As a function of the interference-plus-noise covariance matrix and the desired signal steering vector, adaptive beamformers are sensitive to model mismatch, such as direction error, imperfect array calibration, source wavefront distortions, and insufficient number of training samples. These model mismatches degrade the performance of adaptive beamformers, especially when the training samples are *contaminated* by the desired signal component. To this end, various robust approaches have been proposed in the past decades to decrease the sensitivity of adaptive beamformers (see, for example, [2, 7, 8] and the references therein).

In general, a higher number of training samples (i.e., snapshots) yield a better adaptive beamformer performance. In an ideal scenario where a high number of snapshots are available to achieve accurate covariance matrix estimation, the resulting adaptive beamformer approaches the optimal one. In practice, however, the number of snapshots is limited and, as a result, there are differences between the sample covariance matrix and its statistical covariance matrix, thus leading to a performance loss in the resulting adaptive beamformer. When the number of signal-free snapshots is higher than twice the number of array sensors, the average performance loss of the sample matrix inversion (SMI) beamformer relative to the optimal value is less than 3 dB [9]. Situations with no desired signal component, nevertheless, rarely occur in practical array observations. In this case, an effective approach to avoid performance degradation of the adaptive beamformer is to reconstruct the interference-plus-noise covariance matrix [10–14]. Although covariance matrix reconstruction-based adaptive beamformers achieve a fast convergence, they still require the number of snapshots to be higher than the number of array sensors.

In the scenario of multiple snapshots, adaptive beamformers are usually implemented in a block manner. Such block-adaptive beamformers collect a block of array observation data, estimate and invert the sample covariance matrix, and update the beamforming weights each time a new block of data is received. In such a block-adaptive operation, the beamformer weight cannot respond to the change of quickly moving targets, especially when the block size is large. On the other hand, the number of required snapshots is usually higher than the number of array sensors, thereby demanding a huge burden on the memory space for a large-size array.

In this paper, we consider an extreme scenario, where only a single snapshot is available for adaptive beamformer design. In few existing adaptive beamformer designs with a single snapshot [15, 16], the signal self-nulling problem has not been solved. By exploiting the sparsity of sources in the spatial domain, we formulate a sparsity-constrained covariance matrix fitting problem for the spatial spectrum distribution, from which the source directions and the associated power can be estimated. Accordingly, we can reconstruct a signal-free interference-plus-noise covariance matrix as a weighted sum of the tensor outer products of the interference steering vectors. As such, the signal self-nulling problem can be effectively avoided in the resulting adaptive beamformer. Neither matrix inversion nor eigen-decomposition is required in the proposed single-snapshot adaptive beamformer design. Simulation results demonstrate that the proposed adaptive beamforming algorithm outperforms the other beamforming algorithms suitable for the single-snapshot application over a wide range of input signal-to-noise ratio (SNR).

II. The Signal Model

Assume a narrowband array consisting of M omni-directional sensors. The array observation vector at time t, $\boldsymbol{x}(t) = [x_1(t), \cdots, x_M(t)]^{\mathrm{T}} \in \mathbb{C}^M$, can be expressed as

$$\boldsymbol{x}(t) = \boldsymbol{x}_s(t) + \boldsymbol{x}_i(t) + \boldsymbol{x}_n(t), \tag{1}$$

where $\boldsymbol{x}_s(t)$, $\boldsymbol{x}_i(t)$, and $\boldsymbol{x}_n(t)$ are statistically independent components of the desired signal, interference, and noise, respectively. Here, $(\cdot)^{\mathrm{T}}$ denotes the transpose. Among them, the desired signal vector $\boldsymbol{x}_s(t)$ has the form of

$$\boldsymbol{x}_s(t) = \boldsymbol{a}_s s(t), \tag{2}$$

where $s(t) \in \mathbb{C}$ is the complex-valued signal waveform, and $\boldsymbol{a}_s \triangleq \boldsymbol{a}(\theta_s) \in \mathbb{C}^M$ is the corresponding signal steering vector associated with direction-of-arrival (DOA) θ_s. The steering vector is a function of the source direction for a given array structure. For example, for a uniform linear array (ULA), the steering vector can be expressed as

$$\boldsymbol{a}(\theta) = \left[1, e^{-\jmath \frac{2\pi}{\lambda} d \sin\theta}, \cdots, e^{-\jmath \frac{2\pi}{\lambda}(M-1) d \sin\theta}\right]^{\mathrm{T}}, \tag{3}$$

where θ is the DOA of the source, $\jmath = \sqrt{-1}$ is the imaginary unit, λ is the wavelength of the narrowband signal, and $d = \lambda/2$ is the interelement spacing of the array. Similarly, the interference vector $\boldsymbol{x}_i(t)$ can be expressed as

$$\boldsymbol{x}_i(t) = \sum_{k=1}^{K} \boldsymbol{a}_{i_k} s_{i_k}(t), \tag{4}$$

where K is the number of interferers, and $\boldsymbol{a}_{i_k} \triangleq \boldsymbol{a}(\theta_{i_k})$ is the steering vector corresponding to the k-th interference waveform $s_{i_k}(t)$ impinging from DOA θ_{i_k}.

III. Adaptive beamformers

The objective of the adaptive beamformer is to design a complex-valued beamforming weight vector $\boldsymbol{w} = [w_1, \cdots, w_M]^{\mathrm{T}} \in \mathbb{C}^M$, such that the beamformer output

$$y(t) = \boldsymbol{w}^{\mathrm{H}} \boldsymbol{x}(t), \tag{5}$$

accurately estimates the desired signal waveform $s(t)$, where $(\cdot)^{\mathrm{H}}$ denotes the Hermitian transpose. The most popular adaptive beamforming criterion is to maximize the beamformer output signal-to-interference-plus-noise ratio (SINR), defined as

$$\text{SINR} = \frac{\sigma_s^2 \left|\boldsymbol{w}^{\mathrm{H}} \boldsymbol{a}_s\right|^2}{\boldsymbol{w}^{\mathrm{H}} \boldsymbol{R}_{i+n} \boldsymbol{w}}, \tag{6}$$

where $\sigma_s^2 \triangleq \mathbb{E}\left[|s(t)|^2\right]$ is the desired signal power, and $\boldsymbol{R}_{i+n} \triangleq \mathbb{E}\left[(\boldsymbol{x}_i(t) + \boldsymbol{x}_n(t))(\boldsymbol{x}_i(t) + \boldsymbol{x}_n(t))^{\mathrm{H}}\right] \in \mathbb{C}^{M \times M}$ is the interference-plus-noise covariance matrix. Here, $\mathbb{E}[\cdot]$ denotes the statistical expectation. The maximum SINR problem is equivalent to the minimum variance distortionless response (MVDR) problem [17]

$$\min_{\boldsymbol{w}} \ \boldsymbol{w}^{\mathrm{H}} \boldsymbol{R}_{i+n} \boldsymbol{w} \ \text{ subject to } \ \boldsymbol{w}^{\mathrm{H}} \boldsymbol{a}_s = 1, \tag{7}$$

whose solution

$$\boldsymbol{w}_{\text{MVDR}} = \frac{\boldsymbol{R}_{i+n}^{-1} \boldsymbol{a}_s}{\boldsymbol{a}_s^{\mathrm{H}} \boldsymbol{R}_{i+n}^{-1} \boldsymbol{a}_s}, \tag{8}$$

is called as the MVDR beamfomer, also referred to as the Capon beamformer.

In practice, because the interference-plus-noise covariance matrix $\boldsymbol{R}_{i+n}$ is unknown, it is usually replaced by the sample covariance matrix

$$\hat{\boldsymbol{R}} = \frac{1}{T} \sum_{t=1}^{T} \boldsymbol{x}(t) \boldsymbol{x}^{\mathrm{H}}(t), \tag{9}$$

where T denotes the number of snapshots. This leads to the SMI beamformer [9]

$$\boldsymbol{w}_{\text{SMI}} = \frac{\hat{\boldsymbol{R}}^{-1} \bar{\boldsymbol{a}}_s}{\bar{\boldsymbol{a}}_s^{\mathrm{H}} \hat{\boldsymbol{R}}^{-1} \bar{\boldsymbol{a}}_s}, \tag{10}$$

where $\bar{\boldsymbol{a}}_s = \boldsymbol{a}(\bar{\theta}_s)$ is the presumed steering vector of the desired signal, which may or may not be accurate.

There are two problems using the sample covariance matrix $\hat{\boldsymbol{R}}$ to design the beamformer. First, whenever there is desired signal component in the array observations $\{\boldsymbol{x}(t), t = 1, \cdots, T\}$, the SMI beamformer is essentially a minimum power distortionless response (MPDR) beamformer rather than the expected MVDR beamformer [2, 10]. It will result in the signal self-nulling problem whenever there is a model mismatch. Second, the number of snapshots required for adaptive beamformer design should be larger than or equal to the number of array sensors, i.e., $T \geq M$, to guarantee the invertibility of the sample covariance matrix.

The first problem can be solved by reconstructing an interference-plus-noise covariance matrix to exclude the effect of the desired signal component [10–14]. Regarding to the second problem, although the diagonal loading technique [18, 19] guarantees the matrix to be invertible, few studies have been conducted to design the adaptive beamformer with less snapshots than the number of array sensors, i.e., $T < M$, especially with only a single snapshot.

IV. Proposed adaptive beamforming algorithm

According to the desired signal and interference expressions (2) and (4), the statistical covariance matrix can be expressed as

$$\boldsymbol{R} = \mathbb{E}\left[\boldsymbol{x}(t) \boldsymbol{x}^{\mathrm{H}}(t)\right] = \boldsymbol{R}_s + \boldsymbol{R}_{i+n}. \tag{11}$$

For uncorrelated interferers and noise, the interference-plus-noise covariance matrix becomes

$$\boldsymbol{R}_{i+n} = \sum_{k=1}^{K} \sigma_{i_k}^2 \boldsymbol{a}_{i_k} \boldsymbol{a}_{i_k}^{\mathrm{H}} + \sigma_n^2 \boldsymbol{I}, \tag{12}$$

where $\sigma_{i_k}^2 = \mathbb{E}\left[|s_{i_k}(t)|^2\right]$ is the power of the k-th interferer, σ_n^2 is the noise power, and $\boldsymbol{I}$ is an identity matrix. It is clear that the interference-plus-noise covariance matrix required for adaptive beamformer design is a function of steering vectors and power of interferers, as well as the noise power.

To estimate the sources' parameters including their directions and power, we formulate a sparsity-constrained covariance matrix fitting problem as

$$\begin{aligned} \min_{\boldsymbol{p}, \sigma_n^2} \quad & \left\| \hat{\boldsymbol{R}} - \boldsymbol{A} \boldsymbol{P} \boldsymbol{A}^{\mathrm{H}} - \sigma_n^2 \boldsymbol{I} \right\|_F \\ \text{subject to} \quad & \|\boldsymbol{p}\|_0 = K + 1, \ \boldsymbol{p} \geq \boldsymbol{0}, \ \sigma_n^2 > 0, \end{aligned} \tag{13}$$

where $\boldsymbol{p} \in \mathbb{R}_{+}^{N}$ is the spatial spectrum distribution on the sample grids of the entire observed spatial domain (e.g., $\{\theta_1, \theta_2, \cdots, \theta_N\}$ with $N \gg M$), $\boldsymbol{P} = \text{diag}(\boldsymbol{p})$ is a diagonal matrix of $\boldsymbol{p}$, $\boldsymbol{A} = [\boldsymbol{a}(\theta_1), \boldsymbol{a}(\theta_2), \cdots, \boldsymbol{a}(\theta_N)] \in \mathbb{C}^{M\times N}$ is the array manifold matrix, and $\|\cdot\|_F$ and $\|\cdot\|_0$ respectively denote the Frobenius norm and the ℓ_0 norm. It is intractable to solve the above optimization problem due to the nonconvex ℓ_0 norm constraint, even when the number of interferers K is *a priori* known.

By using the ℓ_1 norm in lieu of the ℓ_0 norm, the nonconvex optimization problem (13) can be relaxed as

$$\begin{aligned} \min_{\boldsymbol{p},\sigma_n^2} \quad & \left\|\hat{\boldsymbol{R}} - \boldsymbol{A}\boldsymbol{P}\boldsymbol{A}^{\mathrm{H}} - \sigma_n^2\boldsymbol{I}\right\|_F \\ \text{subject to} \quad & \|\boldsymbol{p}\|_1 \leq \sigma_s^2 + \sum_{k=1}^{K}\sigma_{i_k}^2 + \sigma_n^2 + \delta, \\ & \boldsymbol{p} \geq \boldsymbol{0},\ \sigma_n^2 > 0, \end{aligned} \tag{14}$$

where the ℓ_1 norm of $\boldsymbol{p}$ equals to the power sum of all sources (i.e., $\sigma_s^2 + \sum_{k=1}^{K}\sigma_{i_k}^2 + \sigma_n^2$). Note that a small number $\delta > 0$ is added to the power constraint in order to allow a space for the optimization algorithm to search for $\boldsymbol{p}$. Although the optimization problem (14) is convex, it is impracticable because it is not easy to accurately estimate the source power in practical applications. Either overestimation or underestimation will sacrifice the solution.

Alternatively, the convex optimization problem (14) can be reformulated as a regularized convex optimization problem

$$\begin{aligned} \min_{\boldsymbol{p},\sigma_n^2} \quad & \left\|\hat{\boldsymbol{R}} - \boldsymbol{A}\boldsymbol{P}\boldsymbol{A}^{\mathrm{H}} - \sigma_n^2\boldsymbol{I}\right\|_F + \gamma\|\boldsymbol{p}\|_1 \\ \text{subject to} \quad & \boldsymbol{p} \geq \boldsymbol{0},\ \sigma_n^2 > 0, \end{aligned} \tag{15}$$

where γ is a regularization parameter that trades off between the sparsity of the estimated spatial spectrum and the covariance matrix fitting error. The reformulated optimization problem (15) intends to find the sparsest spatial spectrum $\boldsymbol{p}$ and the noise power σ_n^2, such that the reconstructed covariance matrix $\boldsymbol{A}\boldsymbol{P}\boldsymbol{A}^{\mathrm{H}} + \sigma_n^2\boldsymbol{I}$ closely approximates the sample covariance matrix $\hat{\boldsymbol{R}}$. This optimization problem is convex, and can be solved using standard and highly efficient interior point method software tools, e.g., CVX [20]. The positions of nonzero entries in the estimated vector $\hat{\boldsymbol{p}}$ represent the estimated DOAs of the desired signal and interferers. Note that there is no matrix inversion or eigen-decomposition required in solving the optimization problem.

Assuming that the known angular sector of $\Theta = [\theta_{min}, \theta_{max}]$ in which the desired signal is located is distinguishable from general locations of the interfering signals, then the peaks in the out-of-sector $\bar{\Theta}$ correspond to the interferers. The interference-plus-noise covariance matrix can be reconstructed as a weighted sum of the tensor outer products of the interference steering vectors as

$$\hat{\boldsymbol{R}}_{i+n} = \sum_{q=1}^{Q}\hat{\boldsymbol{p}}_q\boldsymbol{a}(\hat{\theta}_q)\boldsymbol{a}^{\mathrm{H}}(\hat{\theta}_q) + \hat{\sigma}_n^2\boldsymbol{I}, \tag{16}$$

where the weighted coefficients $\{\hat{\boldsymbol{p}}_q, q = 1, \cdots, Q\}$ are the values of the peaks of the estimated spatial spectrum in the out-of-sector $\bar{\Theta}$, $\{\boldsymbol{a}(\hat{\theta}_q), q = 1, \cdots, Q\}$ are the interference steering vectors with the estimated directions of interferers $\{\hat{\theta}_q, q = 1, \cdots, Q\}$ corresponding to the peaks in $\bar{\Theta}$, and $\hat{\sigma}_n^2$ is the estimated noise power. Here, Q is the number of peaks of the estimated spatial spectrum $\hat{\boldsymbol{p}}$ in the out-of-sector $\bar{\Theta}$.

Because of the introduced convex relaxation (i.e., ℓ_1 norm) and the user-defined regularization parameter (i.e., γ), the spatial spectrum estimated from the regularized convex optimization problem (15) may contain spurious peaks, i.e., there may be more than one peak in the desired signal sector Θ. In such a case, the peak with the highest level is regarded as the estimate of the desired signal, whose corresponding direction is denoted as $\hat{\theta}_s$. On the other hand, if there is no peak in the sector Θ (e.g., when the signal power is weak enough), the adaptive beamformer design uses the assumed signal direction, i.e., $\hat{\theta}_s = \bar{\theta}_s$.

Substituting the reconstructed interference-plus-noise covariance matrix $\hat{\boldsymbol{R}}_{i+n}$ and the estimated signal steering vector $\boldsymbol{a}(\hat{\theta}_s)$ into the MVDR beamformer (8), the proposed adaptive beamformer is given as

$$\boldsymbol{w}_{\text{Pro}} = \frac{\hat{\boldsymbol{R}}_{i+n}^{-1}\boldsymbol{a}(\hat{\theta}_s)}{\boldsymbol{a}^{\mathrm{H}}(\hat{\theta}_s)\hat{\boldsymbol{R}}_{i+n}^{-1}\boldsymbol{a}(\hat{\theta}_s)}, \tag{17}$$

where the reconstructed interference-plus-noise covariance matrix $\hat{\boldsymbol{R}}_{i+n}$ is full-rank and invertible.

In summary, the proposed adaptive beamformer performs the following steps:

Step 1: Compute the sample covariance matrix $\hat{\boldsymbol{R}}$ (9);
Step 2: Solve the sparsity-constrained covariance matrix fitting problem (15);
Step 3: Reconstruct the interference-plus-noise covariance matrix $\hat{\boldsymbol{R}}_{i+n}$ (16);
Step 4: Compute the beamformer weight vector $\boldsymbol{w}_{\text{Pro}}$ (17).

V. Simulation results

In the simulation, a ULA with $M = 10$ omnidirectional sensors spaced a half wavelength apart is used. The desired signal, interference and noise are all modeled as a complex circularly symmetric Gaussian zero-mean spatially and temporally white process. Two interferers are assumed to impinge on the array from directions $-50°$ and $-20°$, respectively. The input interference-to-noise ratio (INR) in each sensor is equal to 30 dB. The desired signal is assumed to be a plane-wave from the presumed direction $\bar{\theta}_s = 5°$. In the performance comparison of the output SINR versus the input SNR, the number of snapshots is fixed to be $T = 1$, namely, a single snapshot. For each scenario, 500 Monte-Carlo trials are performed.

The proposed adaptive beamformer is compared to the delay-and-sum (DAS) beamformer and the diagonal loading SMI (DL-SMI) adaptive beamformer. Note that the data-

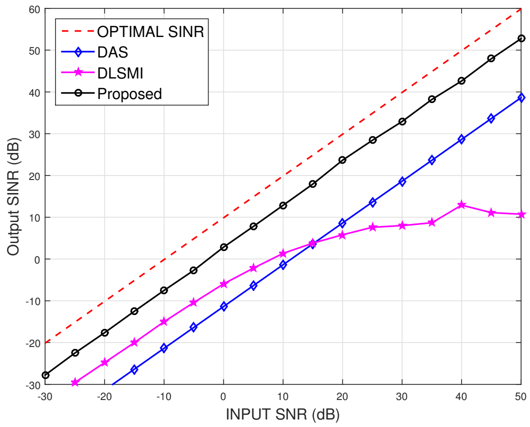

Fig. 1. First example: Fixed look direction mismatch.

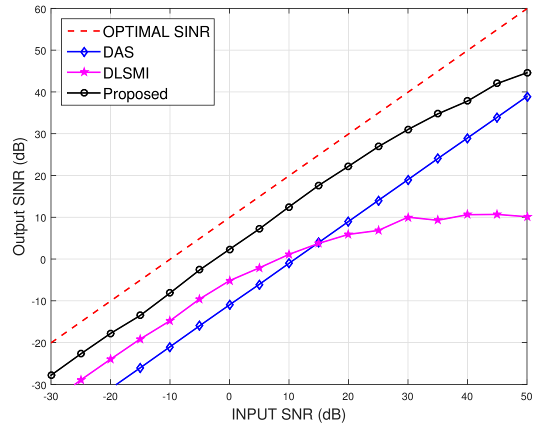

Fig. 2. Second example: Coherent local scattering.

independent DAS beamformer $\boldsymbol{w}_{\text{DAS}} = \frac{\bar{\boldsymbol{a}}_s}{M}$ is not an adaptive beamformer. In the DL-SMI adaptive beamformer

$$\boldsymbol{w}_{\text{DL-SMI}} = \frac{\left(\hat{\boldsymbol{R}} + \varepsilon \boldsymbol{I}\right)^{-1} \bar{\boldsymbol{a}}_s}{\bar{\boldsymbol{a}}_s^{\mathrm{H}} \left(\hat{\boldsymbol{R}} + \varepsilon \boldsymbol{I}\right)^{-1} \bar{\boldsymbol{a}}_s}, \tag{18}$$

the diagonal loading factor is taken as ten times the noise power (i.e., $\varepsilon = 10\sigma_n^2$), where the noise power σ_n^2 is assumed known *a priori*. In the proposed beamformer, the regularization parameter γ is set to be 0.25, and the angular sector where the desired signal is located is set to be $\Theta = [\bar{\theta}_s - 5^\circ, \bar{\theta}_s - 5^\circ]$, namely, $[0^\circ, 10^\circ]$. Other popular adaptive beamformers are not suitable for the considered single-snapshot case because either an inversion or eigen-decomposition of the sample covariance matrix is required there, whereas the single-snapshot observation only provides a rank-one sample covariance matrix. As a benchmark, the optimal output SINR, which is calculated with the known desired signal steering vector and interference-plus-noise covariance matrix, is also shown in all figures.

In the first example, we consider the case where there is a fixed look direction mismatch for the desired signal. The actual signal is assumed to impinge on the array from the direction of $\theta_s = 8^\circ$, which corresponds to a 3° mismatch in the signal look direction. Fig. 1 compares the output SINRs of different beamforming algorithms versus the input SNR. It is clear that, benefiting from the interference-plus-noise covariance matrix reconstruction, the proposed adaptive beamformer outperforms other tested beamformers regardless of the input SNR. For the proposed beamformer, there is about 7 dB performance loss regardless of the desired signal power. This performance loss is due to the inaccurate estimation from the single-snapshot sample. As shown in [10, 11], such performance loss can be reduced to a negligible level as more snapshots are available for adaptive beamformer design. In contrast, the data-independent DAS beamformer uses the presumed beamforming weight vector $\boldsymbol{w}_{\text{DAS}} = \frac{\boldsymbol{a}(5^\circ)}{M}$ rather than the actual beamforming weight vector $\frac{\boldsymbol{a}(8^\circ)}{M}$, which leads to a fixed performance loss of 21 dB regardless of the input SNR. The performance degradation of the DL-SMI adaptive beamformer at high SNR is the self-nulling phenomenon. The reason is that with the increase of SNR, the diagonal loaded covariance matrix $\hat{\boldsymbol{R}} + \varepsilon \boldsymbol{I}$ deviates from the interference-plus-noise covariance matrix $\boldsymbol{R}_{i+n}$ more.

In the second example, we consider a scenario where the desired signal steering vector is distorted by local scattering effects. More specifically, the actual steering vector $\boldsymbol{a}_s$ of the desired signal is assumed to be formed by five signal paths as

$$\boldsymbol{a}_s = \bar{\boldsymbol{a}}_s + \sum_{l=1}^{4} e^{\jmath \psi_l} \boldsymbol{a}(\theta_l), \tag{19}$$

where $\boldsymbol{a}(\theta_l), l = 1, 2, 3, 4$, correspond to the coherently scattered multipaths. The l-th path is modeled as a plane wave with the direction following a normal distribution $\theta_l \sim \mathcal{N}(\bar{\theta}_s, (4^\circ)^2)$ and the phase following a uniform distribution $\psi_l \sim \mathcal{U}[0, 2\pi]$. It should be pointed out that both θ_l and ψ_l change from run to run. It can be seen from Fig. 2 that the proposed adaptive beamformer again outperforms the other tested beamformers, although there is some performance loss in the high SNR region.

VI. Conclusion

In this paper, we have considered adaptive beamformer design with only a single snapshot available. By exploiting the sparsity of the sources, we formulate a sparsity-constrained covariance matrix fitting problem to estimate the parameters required in the reconstruction of the interference-plus-noise covariance matrix for adaptive beamformer design. In such a way, both inversion and eigen-decomposition of the sample covariance matrix is effectively avoided. The proposed adaptive beamformer offers better output SINR performance than existing beamformers applicable for the single-snapshot scenario. Considering that only a single snapshot is required for adaptive beamformer design, the proposed adaptive beamforming algorithm works in a pipeline way rather than in a block way. It is especially attractive for those applications required for a quick response, while the array equips with a very large number of sensors.

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